

Reverse order law for the inverse along an element

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Abstract

In this paper, we introduce a new concept called left (right) g-MP inverse in a $*$ -monoid. The relations of this type of generalized inverse with left inverse along an element are investigated. Also, the reverse order law for the inverse along an element is studied. Then, the existence criteria and formulae of the inverse of the product of triple elements along an element are investigated in a monoid. Finally, we further study left and right g-MP inverses, the inverse along an element in the context of rings.

Keywords:

(Left, Right) Inverses along an element, Reverse order law, Green's preorders, Semigroups, Rings

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1. Introduction

There are many types of generalized inverses in mathematical literature, such as group inverses, Drazin inverses [1], Moore-Penrose inverses [7], (left, right) inverses along an element [3, 8, 9] and so on. Many properties of these generalized inverses were considered in different settings. In particular, a large amount of work has been devoted to the study of the reverse order law for group inverses and Moore-Penrose inverses. However, few results

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have been presented concerning the reverse order law for the inverse along an element since it was introduced.

Throughout this paper, S is a monoid. Let $*$ be an involution on S , that is the involution $*$ satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. We call S a $*$ -monoid if there exists an involution on S .

In this article, we introduce a new concept called left (right) g-MP inverse in a $*$ -monoid S . An element $a \in S$ is called left (resp., right) g-MP invertible if $Sa = Sa^2 = Saa^*a$ (resp., $aS = a^2S = aa^*aS$). The relations of this type of generalized inverse with left inverse along an element will be considered in a monoid. Also, the reverse order law for the inverse along an element is studied. Then, the existence criteria and formulae of the inverse of the product of triple elements along an element are investigated. Finally, we further study left and right g-MP inverses, the inverse along an element in rings.

An element a in S is called (von Neumann) regular if there exists $x \in S$ such that $a = axa$. Such x is called an inner inverse of a , and is denoted by $a^{(1)}$. We call $a \in S$ left (resp., right) regular if $a \in Sa^2$ (resp., $a \in a^2S$). The symbol $a\{1\}$ means the set of all inner inverses of $a \in S$.

Recall that an element $a \in S$ (with involution $*$) is Moore-Penrose invertible (see [7]) if there exists $x \in S$ satisfying the following equations

$$(i) \ axa = a \quad (ii) \ xax = x \quad (iii) \ (ax)^* = ax \quad (iv) \ (xa)^* = xa.$$

Any element x satisfying the equations above is called a Moore-Penrose inverse of a . If such x exists, then it is unique and is denoted by a^\dagger . If x satisfies the equations (i) and (iii), then x is called a $\{1, 3\}$ -inverse of a , and is denoted by $a^{(1,3)}$. If x satisfies the equations (i) and (iv), then x is called a $\{1, 4\}$ -inverse of a , and is denoted by $a^{(1,4)}$. Recall that a^\dagger exists if and only if both $a^{(1,3)}$ and $a^{(1,4)}$ exist. In this case, $a^\dagger = a^{(1,4)}aa^{(1,3)}$. If x in equations (i) and (ii) satisfies $ax = xa$, then a is group invertible. Moreover, the group inverse of a is unique if it exists, and is denoted by $a^\#$. It is known that $a \in S$ is group invertible if and only if it is both left and right regular. We recall that an element $a \in S$ is EP if $a \in S^\# \cap S^\dagger$ and $a^\# = a^\dagger$. The symbols S^\dagger and $S^\#$ denote the sets of all Moore-Penrose invertible and group invertible elements in S , respectively. A well-known characterization of EP elements is that a is EP if and only if $a \in S^\dagger$ and $aa^\dagger = a^\dagger a$.

Green's preorders [2] in S are defined by: (i) $a \leq_{\mathcal{L}} b \Leftrightarrow Sa \subseteq Sb \Leftrightarrow a = xb$ for some $x \in S$; (ii) $a \leq_{\mathcal{R}} b \Leftrightarrow aS \subseteq bS \Leftrightarrow a = by$ for some $y \in S$; (iii) $a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$. Let $a, b, d \in S$. An element b is a left

(resp., right) inverse of a along d [8] if $bad = d$ (resp., $dab = b$) and $b \leq_{\mathcal{L}} d$ (resp., $b \leq_{\mathcal{R}} d$). By $a_l^{\parallel d}$ and $a_r^{\parallel d}$ we denote a left and a right inverse of a along d , respectively. We say that a is invertible along d [3] if there exists b in S such that $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. If such b exists then it is called the inverse of a along d . Moreover, it is unique and is denoted by $a^{\parallel d}$. It is known [8] that a is both left and right invertible along d if and only if it is invertible along d if and only if $d \leq_{\mathcal{H}} dad$.

2. Left (right) g-MP inverses and reverse order law

We begin this section with the following definition, whose properties and relations with left inverse along an element are one of the main study targets in this paper.

Definition 2.1. *Let S be a $*$ -monoid and let $a \in S$. We call a left g-MP invertible if $Sa = Sa^2 = Saa^*a$.*

We next give several examples of left g-MP invertible elements.

Example 2.2. (i) The unity 1 in a $*$ -monoid is left g-MP invertible.

(ii) An EP element a in a $*$ -monoid S is left g-MP invertible. Indeed, $a = a^{\#}a^2 \in Sa^2$, i.e., $Sa = Sa^2$. Also, $a = aa^{\dagger}a = (a^{\dagger})^*a^*a = (a^{\dagger})^*a^{\dagger}aa^*a \in Saa^*a$. So, $Sa = Sa^2 = Saa^*a$.

(iii) Let $S = M_2(\mathbb{C})$ be the monoid of 2×2 complex matrices and let the involution be the conjugate transpose. If $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in S$, then A is left g-MP invertible since $A = I \cdot A^2 = \frac{1}{2}I \cdot AA^*A$.

Next, we give the definition of right g-MP inverse in a $*$ -monoid.

Definition 2.3. *Let S be a $*$ -monoid and let $a \in S$. We call a right g-MP invertible if $aS = a^2S = aa^*aS$.*

Lemma 2.4. [8, Theorem 2.3] *Let $a, d \in S$. Then*

- (i) *a is left invertible along d if and only if $d \leq_{\mathcal{L}} dad$.*
- (ii) *a is right invertible along d if and only if $d \leq_{\mathcal{R}} dad$.*

The following theorem characterizes the relations between $Sa = Saa^*a$ and left inverse along an element in a $*$ -monoid.

Theorem 2.5. *Let S be a $*$ -monoid and let $a \in S$. Then following conditions are equivalent:*

- (i) $Sa = Saa^*a$.
- (ii) a^* is left invertible along a .

PROOF. (i) \Rightarrow (ii) Suppose $Sa = Saa^*a$. Then $a \leq_{\mathcal{L}} aa^*a$. Hence, a^* is left invertible along a by Lemma 2.4.

(ii) \Rightarrow (i) If a^* is left invertible along a , it follows from Lemma 2.4 that $a \leq_{\mathcal{L}} aa^*a$. So, $Sa = Saa^*a$. \square

Dually, we have the following result.

Theorem 2.6. *Let S be a $*$ -monoid and let $a \in S$. Then following conditions are equivalent:*

- (i) $aS = aa^*aS$.
- (ii) a^* is right invertible along a .

It follows from [8, Theorem 2.16] that $aS = aa^*aS$ if and only if $Sa = Saa^*a$ if and only if a is Moore-Penrose invertible. Hence, we get

Corollary 2.7. *Let S be a $*$ -monoid and let $a \in S$. Then following conditions are equivalent:*

- (i) a is Moore-Penrose invertible.
- (ii) a^* is left invertible along a .
- (iii) a^* is right invertible along a .

In this case, $(a^\dagger)^$ is a left (right) inverse of a^* along a .*

PROOF. For the convenience, we only prove that $(a^\dagger)^*$ is a left inverse of a^* along a . As $(a^\dagger)^*a^*a = (aa^\dagger)^*a = aa^\dagger a = a$ and $(a^\dagger)^* = (a^\dagger aa^\dagger)^* = (a^\dagger)^*a^\dagger a \leq_{\mathcal{L}} a$, then $(a^\dagger)^*$ is a left inverse of a^* along a . \square

According to [3, Theorem 11] and Corollary 2.7, $a \in S^\dagger$ if and only if a is invertible along a^* if and only if a^* is invertible along a . Moreover, $a^{\parallel a^*} = a^\dagger$ and $(a^*)^{\parallel a} = (a^\dagger)^*$. We also remark that $a \in S^\#$ if and only if a is invertible along a if and only if 1 is invertible along a . In this case, $a^{\parallel a} = a^\#$ and $1^{\parallel a} = aa^\#$.

From Corollary 2.7, we get that a is left g-MP invertible if and only if it is left regular and Moore-Penrose invertible. If $a \in S$ is both left and right g-MP invertible, then $a \in S^\# \cap S^\dagger$.

The relations between left g-MP inverse and the recently introduced notion called left inverse along an element will be given in Theorem 2.10 below. Herein, we first give relations between $\{1,3\}$ -inverses and $\{1,4\}$ -inverses of an element in a $*$ -monoid.

Lemma 2.8. *Let S be a $*$ -monoid and let $a \in S$. If $a = xaa^*a$ for some $x \in S$, then $(xa)^*$ is both a $\{1,3\}$ -inverse and a $\{1,4\}$ -inverse of a and $a^\dagger = (xa)^*a(xa)^*$.*

PROOF. According to [10, Lemma 2.2], we know that $(xa)^*$ is a $\{1,3\}$ -inverse of a .

Note that

$$\begin{aligned} (xa)^*a &= a^*x^*a = (xaa^*a)^*x^*a = a^*aa^*(x^*)^2a \\ &= a^*(xaa^*a)a^*(x^*)^2a = a^*x(xaa^*a)a^*aa^*(x^*)^2a \\ &= a^*x^2(aa^*)^3(x^2)^*a. \end{aligned}$$

Hence, $[(xa)^*a]^* = (xa)^*a$, that is, $(xa)^*$ is a $\{1,4\}$ -inverse of a . Hence, $a^\dagger = a^{(1,4)}aa^{(1,3)} = (xa)^*a(xa)^*$. \square

Lemma 2.9. *Let S be a $*$ -monoid and let $a \in S$. If $a = aa^*ay$ for some $y \in S$, then $(ay)^*$ is both a $\{1,3\}$ -inverse and a $\{1,4\}$ -inverse of a and $a^\dagger = (ay)^*a(ay)^*$.*

It follows from Lemma 2.4 that $Sa = Sa^2$ if and only if a is left invertible along a . Also, Theorem 2.5 ensures that $Sa = Saa^*a$ if and only if a^* is left invertible along a . Suppose $Sa = Sa^2 = Saa^*a$. Then there exist $x, y \in S$ such that $a = xa^2 = yaa^*a$. So, $a = y(xa^2)a^*a$, which means $a \leq_{\mathcal{L}} a^2a^*a$ and hence aa^* is left invertible along a from Lemma 2.4. One may guess whether the converse holds? that is, if aa^* is left invertible along a implies $Sa = Sa^2 = Saa^*a$? Theorem 2.10 below illustrates this fact.

Theorem 2.10. *Let S be a $*$ -monoid and let $a \in S$. Then the following conditions are equivalent:*

- (i) a is left g-MP invertible.
- (ii) aa^* is left invertible along a .
- (iii) $baa^*a = a$ and $Sb \subseteq Sa$ for some $b \in S$.

*In this case, yxa and baa^*b are left inverses of aa^* along a , for all x and y such that $a = xa^2 = yaa^*a$.*

PROOF. (i) \Rightarrow (ii) It is proved.

(ii) \Rightarrow (i) Suppose that aa^* is left invertible along a . It follows from Lemma 2.4 that $a \leq_{\mathcal{L}} a^2a^*a$, which leads to $Sa = Saa^*a$.

Also, $a \leq_{\mathcal{L}} a^2a^*a$ means $a = ma^2a^*a$ for some $m \in S$. Moreover, $(ma^2)^*$ is a $\{1,4\}$ -inverse of a from Lemma 2.8. Therefore, $a = a(ma^2)^*a = a[(ma^2)^*a]^* = aa^*ma^2$, which yields $Sa = Sa^2$. So, a is left g-MP invertible.

(i) \Rightarrow (iii) Suppose that a is left g-MP invertible. Then there exist $x, y \in S$ such that $a = xa^2 = yaa^*a$. Hence, $a = yxa^2a^*a$. Take $b = yxa$. Then $baa^*a = a$ and $Sb \subseteq Sa$.

(iii) \Rightarrow (i) From $Sb \subseteq Sa$, it follows that $b = na$ for some $n \in S$. Hence, $a = na^2a^*a$. Applying Lemma 2.8, we know that $(na^2)^*$ is a $\{1,4\}$ -inverse of a . So, $a = a(na^2)^*a = a[(na^2)^*a]^* = aa^*na^2$, which implies $Sa = Sa^2$. Also, $baa^*a = a$ can conclude $Sa = Saa^*a$. Therefore, a is left g-MP invertible.

We know that yxa is a left inverse of aa^* along a for all x and y such that $a = xa^2 = yaa^*a$. Indeed, $(yxa)aa^*a = y(xa^2)a^*a = yaa^*a = a$ and $yxa \leq_{\mathcal{L}} a$.

Similarly, one can check that baa^*b is a left inverse of aa^* along a . \square

Dually, we obtain

Theorem 2.11. *Let S be a $*$ -monoid and let $a \in S$. Then the following conditions are equivalent:*

- (i) a is right g-MP invertible.
- (ii) a^*a is right invertible along a .
- (iii) $aa^*ac = a$ and $cS \subseteq aS$ for some $c \in S$.

*In this case, ast and ca^*ac are right inverses of a^*a along a , for all s and t such that $a = a^2s = aa^*at$.*

We next consider some relations among left and right g-MP invertibilities and EP elements.

Theorem 2.12. *Let S be a $*$ -monoid and let $a \in S$. Then the following conditions are equivalent:*

- (i) a is EP.
- (ii) a is left g-MP invertible and $aS = a^*S$.
- (iii) a is right g-MP invertible and $aS = a^*S$.

PROOF. (ii) \Rightarrow (iii) Suppose that a is left g-MP invertible. Then a is Moore-Penrose invertible. Hence, $aS = aa^*(a^\dagger)^*S \subseteq aa^*S = a^2S$ since $aS = a^*S$. Therefore, a is right g-MP invertible.

(iii) \Rightarrow (ii) By noting that $aS = a^*S$ implies $Sa = Sa^*$.

Note that the condition (ii) or (iii) implies $a \in S^\#$. The result follows. Indeed, from [6, Corollary 3], it is known that a is EP if and only if $aS = a^*S$ and $a \in S^\#$ if and only if $aS = a^*S$ and $a \in S^\dagger$. \square

It is well known that the reverse order law holds for the classical inverse in S . More precisely, $(ab)^{-1} = b^{-1}a^{-1}$ for any invertible elements a and b in S . However, $(ab)^{\parallel d} = b^{\parallel d}a^{\parallel d}$ does not hold in general in S . For instance, in the semigroup of 2 by 2 complex matrices, take $a^2 = a = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then $a^{\parallel d} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$. However, $(a^2)^{\parallel d} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{pmatrix} = a^{\parallel d}a^{\parallel d}$.

We remark that a is invertible along d and $ad = da$ imply that d is group invertible in S . Indeed, it is known that $a^{\parallel d}$ exists if and only if $d \leq_{\mathcal{H}} dad$. As $ad = da$, then $d \leq_{\mathcal{H}} dad$ implies $d \in d^2S \cap Sd^2$. Hence, $d \in S^\#$.

Next, we consider the reverse order law for the inverse along an element under commutativity condition.

Lemma 2.13. [3, Theorem 10] *Let $a, d \in S$ with $ad = da$. If $a^{\parallel d}$ exists, then $a^{\parallel d}$ commutes with a and d .*

Theorem 2.14. *Let $a, b, d \in S$ with $ad = da$. If $a^{\parallel d}$ and $b^{\parallel d}$ exist, then*

- (i) $(ab)^{\parallel d}$ exists and $(ab)^{\parallel d} = b^{\parallel d}a^{\parallel d}$.
- (ii) $(ba)^{\parallel d}$ exists and $(ba)^{\parallel d} = a^{\parallel d}b^{\parallel d}$.

PROOF. (i) Since $b^{\parallel d}$ can be written as yd for some $y \in S$, we have

$$b^{\parallel d}a^{\parallel d}abd = b^{\parallel d}aa^{\parallel d}bd = y(daa^{\parallel d})bd = ydbd = b^{\parallel d}bd = d \text{ and } dabb^{\parallel d}a^{\parallel d} = adbb^{\parallel d}a^{\parallel d} = ada^{\parallel d} = daa^{\parallel d} = d.$$

As $a^{\parallel d} \leq_{\mathcal{H}} d$ and $b^{\parallel d} \leq_{\mathcal{H}} d$, then $b^{\parallel d}a^{\parallel d} \leq_{\mathcal{H}} d$.

Therefore, ab is invertible along d and $(ab)^{\parallel d} = b^{\parallel d}a^{\parallel d}$.

(ii) can be proved similarly. \square

We next consider the existence criteria and formulae of left inverse of the product of triple elements along an element.

Theorem 2.15. *Let $a, b, d \in S$. If a is left invertible along d , then the following conditions are equivalent:*

- (i) b is left invertible along d .
- (ii) adb is left invertible along d .

In this case, $(adb)_l^{\parallel d} = b_l^{\parallel d}bya_l^{\parallel d}$, where $y \in S$ satisfies $d = ydbd$.

PROOF. (i) \Rightarrow (ii) Suppose that b is left invertible along d . Then $d \leq_{\mathcal{L}} dbd$ by Lemma 2.4, i.e., $d = ydbd$ for some $y \in S$. Also, as a is left invertible along d , then $d = xdad$ for some $x \in S$. Hence, $d = y(xdad)bd$ and so $d \leq_{\mathcal{L}} dadbd$. Therefore, adb is left invertible along d from Lemma 2.4.

(ii) \Rightarrow (i) As adb is left invertible along d , then $d \leq_{\mathcal{L}} d(adb)d$, which implies $d \leq_{\mathcal{L}} dbd$. Again, Lemma 2.4 guarantees that b is left invertible along d .

We next show that $b_l^{\parallel d} b y a_l^{\parallel d}$ is a left inverse of adb along d for all $y \in S$ such that $d = ydbd$. Indeed, we have $(b_l^{\parallel d} b y a_l^{\parallel d})adb = b_l^{\parallel d} b y d b d = b_l^{\parallel d} b d = d$ and $b_l^{\parallel d} b y a_l^{\parallel d} \leq_{\mathcal{L}} d$ since $a_l^{\parallel d} \leq_{\mathcal{L}} d$. \square

Dually, it follows that

Theorem 2.16. *Let $a, b, d \in S$. If a is right invertible along d , then the following conditions are equivalent:*

- (i) b is right invertible along d .
- (ii) bda is right invertible along d .

In this case, $(bda)_r^{\parallel d} = a_r^{\parallel d} t b b_r^{\parallel d}$, where $t \in S$ satisfies $d = dbdt$.

Next, we present two lemmas which play an important role in the proof of Theorem 2.19 below, which gives equivalences among the inverses of b , adb and bda along d , under certain conditions. The symbol S^{-1} denotes the set of all invertible elements in S .

Lemma 2.17. *Let $e^2 = e \in S$ and let $a \in S$. Then a is invertible along e if and only if $ea e \in (eSe)^{-1}$.*

PROOF. It is known that a is invertible along e if and only if $Se = Seae$ and $eS = eaeS$ if and only if $eSe = eSe(eae) = (eae)eSe$ if and only if $ea e \in (eSe)^{-1}$. \square

Lemma 2.18. *Let $a, d \in S$ and let d be regular with an inner inverse $d^{(1)}$. Then the following conditions are equivalent:*

- (i) a is invertible along d .
- (ii) da is invertible along $dd^{(1)}$.
- (iii) ad is invertible along $d^{(1)}d$.

PROOF. (i) \Rightarrow (ii) Suppose that a is invertible along d . Then $dS = dadS$ and $Sd = Sdad$. As $dS = dd^{(1)}S$, then it follows that $dd^{(1)}S = dadd^{(1)}S = dd^{(1)}dadd^{(1)}S$ and $Sdd^{(1)} = Sdadd^{(1)} = Sdd^{(1)}dadd^{(1)}$. So, da is invertible along $dd^{(1)}$.

(ii) \Rightarrow (i) As da is invertible along $dd^{(1)}$, then $dd^{(1)}S = dd^{(1)}dadd^{(1)}S = dadd^{(1)}S$ and $Sdd^{(1)} = Sdd^{(1)}dadd^{(1)} = Sdadd^{(1)}$. From $dS = dd^{(1)}S$, it follows that $dS = dadS$ and $Sd = Sdadd^{(1)}d = Sdad$, i.e., a is invertible along d .

(i) \Leftrightarrow (iii) It is similar to the proof of (i) \Leftrightarrow (ii). \square

Theorem 2.19. *Let $a, b, d \in S$. If a is invertible along d , then the following conditions are equivalent:*

- (i) b is invertible along d .
- (ii) adb is invertible along d .
- (iii) bda is invertible along d .

In this case, $(adb)^{\parallel d} = b^{\parallel d}d^{(1)}a^{\parallel d}$ and $(bda)^{\parallel d} = a^{\parallel d}d^{(1)}b^{\parallel d}$ for all choices $d^{(1)} \in d\{1\}$.

PROOF. It is known that a is invertible along d implies that d is regular (see [4]). Let $d^{(1)}$ be an inner inverse of d and let $e = dd^{(1)}$ and $f = d^{(1)}d$. As a is invertible along d , then $e(da)e \in (eSe)^{-1}$ and $f(ad)f \in (fSf)^{-1}$ by Lemmas 2.17 and 2.18.

(i) \Leftrightarrow (ii) It follows from Lemmas 2.17 and 2.18 that b is invertible along d if and only if $e(db)e \in (eSe)^{-1}$, and adb is invertible along d if and only if $e(dadb)e \in (eSe)^{-1}$. Note that $e(dadb)e = e(da)e \cdot e(db)e$ and $e(da)e \in (eSe)^{-1}$. Hence, $e(dadb)e \in (eSe)^{-1}$ if and only if $e(db)e \in (eSe)^{-1}$.

(i) \Leftrightarrow (iii) It is similar to (i) \Leftrightarrow (ii) by noting $f(bdad)f = f(bd)f \cdot f(ad)f$.

We next show that $m = b^{\parallel d}d^{(1)}a^{\parallel d}$ is the inverse of adb along d .

We have $a^{\parallel d}ad = d = daa^{\parallel d}$ and $a^{\parallel d} = x_1d = dx_2$ for some $x_1, x_2 \in S$. Also, $b^{\parallel d}bd = d = dbb^{\parallel d}$ and $b^{\parallel d} = y_1d = dy_2$ for some $y_1, y_2 \in S$.

It follows that

$$\begin{aligned} madbd &= b^{\parallel d}d^{(1)}(a^{\parallel d}ad)bd = b^{\parallel d}d^{(1)}dbd \\ &= y_1dd^{(1)}dbd = y_1dbd = b^{\parallel d}bd \\ &= d \end{aligned}$$

and

$$\begin{aligned}
dadb m &= da(db b^{\parallel d})d^{(1)}a^{\parallel d} = dadd^{(1)}a^{\parallel d} \\
&= dadd^{(1)}dx_2 = dadx_2 = daa^{\parallel d} \\
&= d.
\end{aligned}$$

As $m = b^{\parallel d}d^{(1)}a^{\parallel d} = dy_2d^{(1)}a^{\parallel d} = b^{\parallel d}d^{(1)}x_1d$, then $m \leq_{\mathcal{H}} d$.

Hence, $(adb)^{\parallel d} = b^{\parallel d}d^{(1)}a^{\parallel d}$.

Similarly, we can check $(bda)^{\parallel d} = a^{\parallel d}d^{(1)}b^{\parallel d}$. \square

Remark 2.20. The assumption “ a is invertible along d ” in Theorem 2.19 can not be dropped. Indeed, take S be the monoid of all infinite complex matrices with finite nonzero elements in each column and let $d = 1 \in S$, $a = \sum_{i=1}^{\infty} e_{i,i+1} \in S$ and $b = \sum_{i=1}^{\infty} e_{i+1,i} \in S$, where $e_{i,j}$ denotes the infinite matrix whose (i,j) -entry is 1 and other entries are zero. Then $1 = adb = ab$ is invertible along 1. However, $\sum_{i=2}^{\infty} e_{i,i} = bda = ba$ is not invertible along 1.

As a consequence, we present a result on the reverse order law for the inverse of the product of triple elements along an element.

Corollary 2.21. *Let $a, b, d \in S$. If a is invertible along d and $d \in S^{\#}$, then the following conditions are equivalent:*

- (i) b is invertible along d .
- (ii) adb is invertible along d .
- (iii) bda is invertible along d .

In this case, $(adb)^{\parallel d} = b^{\parallel d}d^{\parallel d}a^{\parallel d}$ and $(bda)^{\parallel d} = a^{\parallel d}d^{\parallel d}b^{\parallel d}$.

As special results of Theorem 2.19, it follows that

Corollary 2.22. *Let $b, d \in S$. If $d \in S^{\#}$, then the following conditions are equivalent:*

- (i) b is invertible along d .
- (ii) bd is invertible along d .
- (iii) db is invertible along d .

In this case, $(bd)^{\parallel d} = d^{\parallel d}b^{\parallel d}$ and $(db)^{\parallel d} = b^{\parallel d}d^{\parallel d}$.

PROOF. Since $1^{\parallel d} = dd^{\#}$, $d^{\parallel d} = d^{\#}$ and $b^{\parallel d} = dx$ for some $x \in S$, we have $(bd)^{\parallel d} = 1^{\parallel d}d^{(1)}b^{\parallel d} = d^{\#}dd^{(1)}dx = d^{\#}dx = d^{\#}b^{\parallel d} = d^{\parallel d}b^{\parallel d}$.

We may use the same reasoning to obtain $(db)^{\parallel d} = b^{\parallel d}d^{\parallel d}$. \square

Suppose $a = d^*$ in Theorem 2.19. Then it follows that

Corollary 2.23. *Let S be a $*$ -monoid and let $b, d \in S$. If $d \in S^\dagger$, then the following conditions are equivalent:*

- (i) b is invertible along d .
- (ii) d^*db is invertible along d .
- (iii) bdd^* is invertible along d .

*In this case, $(d^*db)^{\parallel d} = b^{\parallel d}d^{\parallel d^*}(d^*)^{\parallel d}$ and $(bdd^*)^{\parallel d} = (d^*)^{\parallel d}d^{\parallel d^*}b^{\parallel d}$.*

PROOF. By Theorem 2.19, we have $(d^*db)^{\parallel d} = b^{\parallel d}d^{(1)}(d^*)^{\parallel d}$. Note that $(d^*)^{\parallel d} = (d^\dagger)^*$ and $b^{\parallel d}$ can be written as xd for an appropriate x in S . Hence, $(d^*db)^{\parallel d} = xdd^{(1)}dd^\dagger(d^\dagger)^* = xdd^\dagger(d^\dagger)^* = b^{\parallel d}d^{\parallel d^*}(d^*)^{\parallel d}$.

Similarly, $(bdd^*)^{\parallel d} = (d^*)^{\parallel d}d^{\parallel d^*}b^{\parallel d}$. □

3. Further results in rings

In this section, let R be an associative unital ring. An involution $*$: $R \rightarrow R$; $a \mapsto a^*$ is an anti-isomorphism on R satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a + b)^* = a^* + b^*$ for all $a, b \in R$.

We next begin with two lemmas, which play an important role in the sequel.

Lemma 3.1. *Let $a, b \in R$. Then we have*

- (i) *If $(1 + ab)x = 1$ for some $x \in R$, then $(1 + ba)(1 - bxa) = 1$.*
- (ii) *If $y(1 + ab) = 1$ for some $y \in R$, then $(1 - bya)(1 + ba) = 1$.*

By the symbols R_l^{-1} , R_r^{-1} and R^{-1} we denote the sets of all left invertible, right invertible and invertible elements in R , respectively. It follows from Lemma 3.1 that $1 + ab \in R_l^{-1}$ if and only if $1 + ba \in R_l^{-1}$, and $1 + ab \in R_r^{-1}$ if and only if $1 + ba \in R_r^{-1}$. Further, $1 + ab \in R^{-1}$ if and only if $1 + ba \in R^{-1}$. In this case, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$, which is known as Jacobson's Lemma.

Lemma 3.2. [8, Corollaries 3.3 and 3.5] *Let $a, m \in R$ with m regular. Then the following conditions are equivalent:*

- (i) a is (left, right) invertible along m .
- (ii) $u = ma + 1 - mm^{(1)}$ is (left, right) invertible.
- (iii) $v = am + 1 - m^{(1)}m$ is (left, right) invertible.

It is known that $1^{\parallel a}$ exists if and only if $a^\#$ exists in a ring R . Hence, Lemma 3.2 also gives an existence criterion of group inverse, that is, a is group invertible if and only if $a + 1 - aa^{(1)}$ is invertible if and only if $a + 1 - a^{(1)}a$ is invertible, for a regular element $a \in R$.

Given an element $a \in R$, we use the symbols a_l^{-1} and a_r^{-1} to denote a left and a right inverse of a , respectively.

Applying Corollary 2.7 and Lemma 3.2, we derive the following result, which recovers the classical existence criterion of Moore-Penrose inverse (see, e.g. [5, Theorem 1.2]) in a ring.

Theorem 3.3. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent:*

- (i) $a \in R^\dagger$.
 - (ii) $u = aa^* + 1 - aa^{(1)}$ is left invertible.
 - (iii) $v = a^*a + 1 - a^{(1)}a$ is left invertible.
 - (iv) $u = aa^* + 1 - aa^{(1)}$ is right invertible.
 - (v) $v = a^*a + 1 - a^{(1)}a$ is right invertible.
- In this case, $a^\dagger = (u_l^{-1}a)^*a(u_l^{-1}a)^* = (av_r^{-1})^*a(av_r^{-1})^*$.

PROOF. As $u = aa^* + 1 - aa^{(1)}$, then $ua = aa^*a$. It follows that $a = u_l^{-1}aa^*a$ since u is left invertible. By Lemma 2.8, we know that $(u_l^{-1}a)^*$ is both a $\{1, 3\}$ -inverse and a $\{1, 4\}$ -inverse of a . Hence, $a^\dagger = a^{(1,4)}aa^{(1,3)} = (u_l^{-1}a)^*a(u_l^{-1}a)^*$.

Similarly, as $v = a^*a + 1 - a^{(1)}a$ is right invertible, then $a = aa^*av_r^{-1}$. Lemma 2.9 ensures $a^\dagger = (av_r^{-1})^*a(av_r^{-1})^*$. \square

As a corollary of Theorem 3.3, it follows that

Corollary 3.4. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent:*

- (i) $a \in R^\dagger$.
 - (ii) $u = aa^* + 1 - aa^{(1)}$ is invertible.
 - (iii) $v = a^*a + 1 - a^{(1)}a$ is invertible.
- In this case, $a^\dagger = (u^{-1}a)^*a(u^{-1}a)^* = (av^{-1})^*a(av^{-1})^*$.

From Theorem 2.10 and Lemma 3.2, it follows that

Corollary 3.5. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent:*

- (i) *a is left g -MP invertible.*
- (ii) *$a^2a^* + 1 - aa^{(1)}$ is left invertible.*
- (iii) *$aa^*a + 1 - a^{(1)}a$ is left invertible.*

By Theorem 2.11 and Lemma 3.2, we get

Corollary 3.6. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent:*

- (i) *a is right g -MP invertible.*
- (ii) *$a^*a^2 + 1 - a^{(1)}a$ is right invertible.*
- (iii) *$aa^*a + 1 - aa^{(1)}$ is right invertible.*

The following theorem gives characterizations of left and right g -MP inverses in terms of units in a ring.

Theorem 3.7. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent:*

- (i) *a is both left and right g -MP invertible.*
- (ii) *$u = aa^*a + 1 - aa^{(1)}$ is invertible.*
- (iii) *$v = aa^*a + 1 - a^{(1)}a$ is invertible.*

PROOF. (i) \Rightarrow (ii) If a is both left and right g -MP invertible, then a is both group invertible and Moore-Penrose invertible. As a is group invertible, then it follows that $a + 1 - aa^{(1)} \in R^{-1}$. Also, a is Moore-Penrose invertible implies that $aa^* + 1 - aa^{(1)} \in R^{-1}$ by Corollary 3.4. Hence, $aa^*aa^{(1)} + 1 - aa^{(1)} \in R^{-1}$ by Jacobson's Lemma. So, $(aa^*aa^{(1)} + 1 - aa^{(1)})(a + 1 - aa^{(1)}) = u \in R^{-1}$.

(ii) \Rightarrow (i) As $u = aa^*a + 1 - aa^{(1)} \in R^{-1}$, then $u' = a^*a^2 + 1 - a^{(1)}a \in R^{-1}$ from Jacobson's Lemma. Hence, $au' = aa^*a^2$ and $a = aa^*a^2(u')^{-1} \in aa^*aR$, which means that a is Moore-Penrose invertible by [8, Theorem 2.16]. Applying Corollary 3.4, we know that if a is Moore-Penrose invertible then $aa^* + 1 - aa^{(1)} \in R^{-1}$ and hence $aa^*aa^{(1)} + 1 - aa^{(1)} \in R^{-1}$ according to Jacobson's Lemma. Note that $a + 1 - aa^{(1)} = (aa^*aa^{(1)} + 1 - aa^{(1)})^{-1}u \in R^{-1}$. It follows that a is group invertible. Thus, a is both left and right g -MP invertible.

(i) \Leftrightarrow (iii) It is similar to (i) \Leftrightarrow (ii). □

Given a regular element $d \in R$, it is known from [4, Theorem 3.2] that $a^{\parallel d}$ exists if and only if $da + 1 - dd^{(1)}$ is invertible if and only if $ad + 1 - d^{(1)}d$ is invertible. Moreover, $a^{\parallel d} = (da + 1 - dd^{(1)})^{-1}d = d(ad + 1 - d^{(1)}d)^{-1}$. Applying this, we have

Corollary 3.8. *Let $a, b, d \in R$ and let a be invertible along d . Then the following conditions are equivalent:*

- (i) *b is invertible along d .*
- (ii) *adb is invertible along d .*
- (iii) *bda is invertible along d .*

In this case, $(adb)^{\parallel d} = b^{\parallel d}v^{-1}$ and $(bda)^{\parallel d} = u^{-1}b^{\parallel d}$, where $u = da + 1 - dd^{(1)}$ and $v = ad + 1 - d^{(1)}d$.

PROOF. It follows from Theorem 2.19 that $(adb)^{\parallel d} = b^{\parallel d}d^{(1)}a^{\parallel d}$. As $a^{\parallel d} = dv^{-1}$ and $b^{\parallel d} = y_1d$ for some $y \in R$, then $(adb)^{\parallel d} = y_1dd^{(1)}dv^{-1} = y_1dv^{-1} = b^{\parallel d}v^{-1}$.

Similarly, $(bda)^{\parallel d} = u^{-1}b^{\parallel d}$. □

Corollary 3.9. *Let $a, b, d \in R$ and let a be invertible along d . Then the following conditions are equivalent:*

- (i) *b is invertible along d .*
- (ii) *$u = dadb + 1 - dd^{(1)}$ is invertible.*

In this case, $b^{\parallel d} = u^{-1}dad$.

PROOF. Since $u = dadb + 1 - dd^{(1)} = (dadd^{(1)} + 1 - dd^{(1)})(db + 1 - dd^{(1)})$, it follows that $db + 1 - dd^{(1)} = (dadd^{(1)} + 1 - dd^{(1)})^{-1}u$. Applying [4, Theorem 3.2], we have $b^{\parallel d} = (db + 1 - dd^{(1)})^{-1}d = u^{-1}(dadd^{(1)} + 1 - dd^{(1)})d = u^{-1}dad$. □

4. Remarks and questions

We close this section with a remark and a question:

4.1. It is known that ab is invertible along d may not imply ba is invertible along d in a monoid (see Remark 2.20). Assume that ab and ba are both invertible along d in a monoid S . Does Cline's formula for the inverse along an element hold? i.e., if $(ab)^{\parallel d} = a((ba)^{\parallel d})^2b$? In fact, this formula does not hold. For instance, let S be the monoid of 2 by 2 complex matrices. Set

$a^2 = a = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in S$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S$ and $d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in S$. Then $a^{\parallel d} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$. But $a((ba)^{\parallel d})^2b = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} = (ab)^{\parallel d}$.

4.2. Suppose that ab and ba are both invertible along d in a monoid. Can we give a necessary and sufficient condition such that Cline's formula for the inverse along an element to hold.

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